Emergence of Power Law in a Market with Mixed Models.

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Abstract

We investigate the problem of wealth distribution from the viewpoint of asset exchange. Robust nature of Pareto's law across economies, ideologies and nations suggests that this could be an outcome of trading strategies. However, the simple asset exchange models fail to reproduce this feature. A yardsale(YS) model in which amount put on the bet is a fraction of minimum of the two players leads to condensation of wealth in hands of some agent while theft and fraud(TF) model in which the amount to be exchanged is a fraction of loser's wealth leads to an exponential distribution of wealth. We show that if we allow few agents to follow a different model than others, *i.e.* there are some agents following TF model while rest follow YS model, it leads to distribution with power law tails. Similar effect is observed when one carries out transactions for a fraction of one's wealth using TF model and for the rest YS model is used. We also observe a power law tail in wealth distribution if we allow the agents to follow either of the models with some probability.

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1 Introduction

Rich get richer and poor get poorer. Worse, rich people do not seem to be significantly cleverer or more hardworking than the poorer lot. This has puzzled philosophers and economists alike all the way from Buddha to Marx. A century ago, an Italian economist Pareto gave a celebrated empirical law suggesting that it is just a law of nature that 80% of the wealth is in 20% hands. In fact, on surveying various countries and economies in Europe, he gave a famous law, now known as Pareto law. It said that the probability $P(x)$ that an individual has wealth x follows a power law for large x, *i.e.* $P(x) \sim x^{-\nu}$. Distribution of personal wealth and income in countries as diverse as USA, UK, Japan and India seem to have a power law tail [1, 2, 3, 4, 5]. Since it seems to be independent of the political systems of those countries, which were widely different, it can be conjectured that this distribution is the inherent outcome of economic

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activities. In fact, it was recently observed that even in ancient Egyptian society, wealth distribution could be following the power law[6]. In this society, obviously myriad of factors playing role in modern economies do not exist. Thus it could be conjectured that there has to be an explanation for the law from simple and primitive principles of economic activity.

One of the striking economic activity that every society, including the most ancient ones, is capable of, is 'give and take'. Someone gives you a cup of coffee, you hand him over a dollar. This is a simple exchange of assets. Let us call it 'additive asset exchange'. However, you do not keep your entire wealth at stake to a coffee shop owner. But you may keep it at stake in bank. Here you get money which is proportionate to the money you own. Let us call it 'multiplicative asset exchange'. Researchers have looked at the models of wealth distribution in presence of additive and multiplicative asset exchange and results are intriguing. Somehow, simple asset exchange models are unable to reproduce the power law tail which seems to be robust feature across economies. For multiplicative asset exchange models, in which all agents start with same wealth, have similar capabilities (none is cleverer than the other in any sense) the emerging distribution of wealth is even less equitable than a power law. It turns out that in 'free and fair' trade, one agent (by pure luck, since we have not assigned extra capabilities to any agent) ends up swallowing the entire wealth. In another 'theft and fraud' rule, we get an exponential distribution of wealth.

What are these models?[7] We consider two models given by Brian Hayes. In these models, there is no consumption of wealth nor any production. In the first model, we assume that everyone knows the value of everybody else's asset perfectly. This 'free and fair' (since nobody is able to conceal true value of his assets) model is called Yardsale model[8]. It is the following: There are N individuals in society and they trade with each other on one-to-one basis. Everyone is able to value everyone else's assets perfectly while trading. Naturally, the amount traded is a fraction of assets of poorer party. However, we can have another rule. In this rule, the amount to be exchanged is a fraction of loser's wealth. Naturally, poorer agents have more to gain by playing with richer ones and they can do so only by deception. Hence it has been named theft and fraud (TF) model[9]. The YS and TF rules can be given as follows. Let us consider set of N agents with wealth $m_1(0), m_2(0) \ldots m_N(0)$ at time $T = 0$. At each timestep $T = t$ we choose two agents i and j and their wealths $m_i(t)$ and $m_j(t)$ are updated as:

$$m_i(t + 1) = m_i(t) + \Delta m \tag{1}$$

$$m_j(t + 1) = m_j(t) - \Delta m \tag{2}$$

where Δm is the net wealth exchanged between that two agents. (Wealth of rest of the agents is unchanged.) In the YS model, $\Delta m = \alpha \min(m_i(t), m_j(t))$. Whereas, in the TF model the money exchange is fraction of the wealth loser

player. Then, $\Delta m = \alpha(m_j(t))$ (if j is the loser). The parameter α is a uniformly distributed random number in the interval $[0, 1]$.

However, none of these models reproduces the power law distribution of wealth found in several societies. The YS model essentially produces condensation of wealth in hands of one of the agents. Whereas, the TF model yields an exponential distribution of assets. None of these models reflect the empirically observed distribution of wealth [10]. To mimic these observed features of income and wealth distribution, several efforts have been made. Some researchers have applied the techniques of statistical physics to the economic system. They treat the economic agents as particles in gas. The total wealth is conserved which is analogous to energy in ideal gas. Thus the equilibrium probability distribution of money $P(x)$ should follow the Boltzmann-Gibbs law $P(x) = c \exp^{-x/T}$. Here x is money, T the average money per agent and c is constant [11] . Chatterjee and Chakrabarti argued that not all the money is put at stake in market. Every economic agent saves something for a rainy day. They studied the effect of saving propensities for the agents [12]. Two cases have been studied. In the first case, all the agents have the same fixed saving factor [12] while in other case the agents have a quenched random distribution of saving factors[13]. The former case yields the gaussian distribution of wealth while the later model gives a power law distribution of wealth. The other model was introduced by Sinha. He assumed that a richer player is less likely to be aggressive when bargaining over a small amount with a poorer player. When this role is added to the YS model we can see the exponential and power-law distribution of wealth [10]. A similar model was also introduced by Rodríquez-Achach and Huerta-Quintanilla[14]. Several researchers obtained Pareto-like behavior using different approaches such as: rich people trading with the gross system while poorer agents continue to have two-party transaction [15], flow of wealth from outside leading to inelastic scattering [16], generalized Lotka Volterra dynamics [17] and stochastic evolution equation which incorporate trading as well as random changes in prices of investments [18].

Here, we would like to argue that entire society playing with a simple model is unlikely and unrealistic. It is quite likely that different players play within different paradigms. Thus it is important to investigate wealth distribution in societies where we have mixed models. In this work, we will study the system from this viewpoint. In one model, we will let a few players to play by TF model while the others to play by YS model. We will study the effect of this mixing on the distribution of wealth at equilibrium. In the second model, the agent will invest some part of his money in YS model and remaining part in TF trade system.

We try yet another system in which players take decision to play using YS or TF model with certain probability. This could be termed as a system with multiple strategies since unlike previous case, players have a choice to play using either of the models.

We note that condensation can not be steady state in the above models.

The richest agent, while playing by TF model or playing with a player using TF model could lose his wealth easily. We observe that few agents using TF strategy would significantly change the possible steady state of the system. All these systems lead to a wealth distributions which show a clear power-law tail at higher values of wealth which is comparable to realistic situations in some cases.

2   The Model(s)

We assume that despite limitations, YS model is the most reasonable model of asset exchange. We argue that the wealth distributions observed in reality are due to perturbations to this model. The perturbations we introduce seem not only be able to deliver a power law tail in the wealth distribution of individuals, but we also get a variety of exponents as seen in realistic economies. We consider a closed economic system where the total amount of money M is conserved and the number of economic agents N is fixed. No debt is permitted. Initially, we divide the total money M among N agent equally.

   We consider three different models. In the first model, we introduce TF agents in a pure YS model. In the second model, we presume that all asset exchanges are partly TF and partly YS. In the third model, we will let each player to choose to use YS or TF model with certain probability. All these variants give us a power-law distribution of wealth.

   (A) We consider a case of one agent playing by TF model (say kth agent) while everyone else plays by YS model. The transaction will go according to the following scheme. We choose any two players i and j randomly. If $i = k$ or $j = k$, the transaction between i and j goes by TF model. (We must note that this asymmetry is not an essential prerequisite of the results. In the case that, if $i = k$ or $j = k$, the transaction goes by TS or YF with equal probability, our results do not change.) If both agents are following YS model, transaction rule is YS. We must mention that even one agent playing with TF model ruins the possibility of condensation of wealth. Thus, the asymptotic distribution is expected to change. We observe that it changes significantly even in presence of one agent.

   (B) We will let all players to use a part of their money $\lambda_i m_i(t)$ in YS strategy and the other part $(1 - \lambda_i)m_i(t)$ to used it in TF trade. In this case, the wealth distribution will depend on distribution of λ_is. We study two cases of distribution of λ_is: (i) λ_i's have same value for all agents, *i.e.* $\lambda_i = \lambda$ for $1 \leq i \leq N$. (ii) λ_i's have a quenched random distribution. Let us consider this to be uniform distribution over an interval $[0, 1]$.

   (C) We consider a case in which every agent can trade by either of models with some probability. We suppose that ith agent has inclination to trade by YS model with probability p_i and by TF model with probability $(1 - p_i)$. We will let each agent to choose the value of p from a uniform random number in

the interval $[0, 1]$. We will assume the quenched state, where the agents have different value of p. We conduct the transactions as follows: we select two agents i and j randomly. The players choose to trade by TF model with probability p_i and p_j and by YS model with probability $1 - p_i$ and $1 - p_j$. If the two agents chose different model the transaction will occur by YS model. (Even here, we must mention that asymmetry does not play a significant role. If we make a rule that transaction will be there *if and only if* both agents follow the same model, we get the same asymptotic distribution.) We observe that the asymptotic wealth distribution has a power law tail with fairly high value of exponent.

3 The Simulation and Result

In these models, we need to find the asymptotic probability distribution. We need to employ certain systematic approach to check if the asymptotic distribution is actually reached. In all these cases we find that the average wealth of the richest agent as an useful quantifier. We plot this quantity as a function of time and have taken the saturation of this quantity as an indicator of the possibility that the wealth distribution has saturated. (Apart from this plot, we have also checked the wealth distribution at different timesteps and have checked if it has converged.)

In model A, we recorded the wealth of the richest agent. We have introduced only one TF agent in a sea of YS agents. The average wealth of richest agent is plotted as a function of time in Fig. 1. Here we notice that average wealth of the richest agent saturates to *same* value for all values of N. This value is not unity. Thus one TF agent is able to qualitatively change the dynamics of the system. Not only that, all these models show similar characteristics, though the effect of single TF agent is apparent in a larger system only after a longer time. Let us denote the which wealth of richest agent saturates as t_c. As expected t_c increases with N. The saturation time t_c scales with N as $t_c(N) \simeq aN^b$ with $a \simeq 0.90$ and $b \simeq 2.23$. This behavior is depicted in Fig. 2. The $t_c(N)$ gives us an idea about the time needed for the system to attain the steady state. We study the wealth distributions for $t > t_c(N)$. In Fig. 3, we show the wealth distribution for N=100, for $t = 10^6$. We average over 3×10^3 initial conditions. The system follows a power-law wealth distribution with exponent $\nu \simeq 1.1$. We checked the robustness of the distribution at various values of N, i. e. $N = 300$, at time $t = 10^6$. We again average over 3×10^3 initial conditions. This distribution also follows a power-law tail with the same exponent $\nu \simeq 1.1$. Pareto exponent in this strategy will be $\simeq 0.1$. It is very small compared to that observed in real economies.

In model B, agents use different fractions of their money in YS and TF models. This could be compared with individuals investing their money in bonds and stocks. When one has bought stocks, one is paid according to performance

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of the company rather than his own wealth at that time. Thus it is a realistic situation in modern context. First, we consider the case in which the fraction λ is a constant, *i.e.* one uses $(1-\lambda)m_i(t)$ with TF model and rest of the money in YS model. We again study the evolution of wealth of the richest agent as a function of time. We find that the wealth of richest agent saturates at certain time and as in previous case, we denote this time by $t_c(N)$. In Fig. 4 we have plotted $t_c(N)$ as a function of N for $\lambda = 0.999$. We can see that the saturation time $t_c(N)$ scales with number of agents as $t_c(N) \simeq aN^b$ with $a \simeq 513$ and $b \simeq 1.204$.

For $\lambda = 1$, we have a pure YS model which leads to condensation of wealth and $\lambda = 0$ we have a TF model which leads to an exponential distribution. For λ very close to 1 but not exactly 1, we observe that the asymptotic distribution has a power law tail. In the general case $0 < \lambda < 1$, as λ increase from 0 to 1, the asymptotic distribution of wealth is observed to go from exponential to condensate. In Fig. 5, for $\lambda = 0.999$. we demonstrate the asymptotic wealth distribution. It clearly displays a power-law with exponent $\simeq 1.5$, The simulation was carried out for $N = 100$, $t = 10^6$ iterations and averaged over 3×10^3 initial conditions. This model has a tunable parameter λ and only for values close to unity we observe a power law behavior.

However, everyone has a different appetite for risk. We attempt a model in which λ_i's have a quenched random distribution. We consider a uniform distribution of λ's. As in previous case, we find the saturation time t_c at which maximum wealth saturates. In Fig. 6, we have plotted $t_c(N)$ as a function of N for this model. The saturation time $t_c(N)$ scales with number of agents as $t_c(N) \simeq aN^b$ with $a \simeq 240$ and $b \simeq 1.561$. It is interesting to note that the steady state of wealth distribution has a power-law tail with $\nu \simeq 2.0$ as that shown in Fig. 7. However, in the region corresponding to low wealth, the wealth distribution is found to be exponential. The inset of Fig. 7 show that behavior. In this strategy Pareto exponent is $\simeq 1.0$. This strategy seem to be more realistic as compared to model A previously discussed and it also gives exponents which are comparable to realistic case.

In model C, the transaction is carried on depending on the agent's choice who chooses to play with YS or TF model with certain probability. Now some transactions will follow YS model and asset exchange in rest of the transactions will be decided by TF model. We define the saturation time $t_c(N)$ by looking at wealth of richest agent as done in previous cases. We observe that $t_c(N) \simeq aN^b$ with $a \simeq 16.82$ and $b \simeq 1.56$ in this case. (See Fig. 8.) The asymptotic distribution of wealth shown in Fig. 9. It has a power-law tail with exponent $\nu \simeq 3.7$. In this case Pareto exponent is $\simeq 2.7$ which is comparable to one observed in societies like Italy.[20]

Table 1: Comparison of Power-law and Lognormal Fits.			
Model	Fit	χ^2/Dof	R^2
Model A	Lognormal	3×10^{-5}	0.4911
	Power-law	7×10^{-7}	0.9985
Model B case a)	Lognormal	10^{-5}	0.57
	Power-law	5×10^{-7}	0.9976
Model B case b)	Lognormal	10^{-5}	0.057
	Power-law	2×10^{-6}	0.8183
Model C	Lognormal	10^{-3}	0.30
	Power-law	10^{-4}	0.87

We have checked that the power law is a better fit than lognormal for all the cases discussed in the paper in several ways. We have checked it visually. We have checked the goodness of fit by finding χ^2/DoF and R^2 for three models by fitting it a power law functional form and lognormal fit. The values are given in Table 1. It is clear that R^2 values are higher and very close to unity for power-law fit which shows that this model is relevant for higher fraction of data and χ^2/DoF values are lower for a power law fit which shows that error is smaller in this fit in all cases.

For models B and C where Pareto exponent is more than one, we have also plotted the Zipf plot. We order the wealth of agents in descending order and plot the wealth of kth ranked agent as a function of its rank k. It is known that if the probability distribution has a tail of nature $P(x) \sim x^{-(1+\alpha)}$, the rank distribution $x_k \sim k^{-1/\alpha}$ where x_k is the kth largest value in the distribution. Our Zipf plots are consistent with this result.

4 Conclusions

We point out that having a society in which all agents use the same model is unrealistic. The agents are likely to use different models. In this context, we studied YS and TF models. For a pure YS model, condensation of wealth is observed while a pure TF model leads to exponential distribution. We have presented three different models in which the above two models are mixed. In model A, we showed that infinitesimal fraction of TF agents can significantly alter wealth distribution of society where dominant model is YS. If we equate YS model with 'honesty', and TF model with 'cheating', the presence of the other possibility seems to help the society to have more equitable distribution though attaching these virtues to these models is debatable[9]. This mixing gives rise to the wealth distribution with power- law behavior with exponent $\nu \simeq 1.1$. In model B, we considered each transaction to be consisting of YS and TF component. It also leads to a power law tail in wealth distribution. This could be thought as individual investing in debt market as well as in real estate or bonds where return is proportional to performance of the company he invested in. Here we considered two cases a) Homogeneous agents where

they put λ fraction of their wealth in YS model. b) Inhomogeneous agents putting λ_i of his wealth (say x_i) in YS model where λ_i have quenched random distribution. In former case, for λ close to one, *i.e.* for $\lambda = .999$ we observe power-law with exponent $\nu \simeq 1.5$. In the later case, we observe that the wealth distribution has a power-law tail with exponent $\nu \simeq 2.0$. Interestingly, we also recover exponential decay at smaller values of wealth which matches with known data about wealth distribution in United Kingdom and United States [21]. We also studied a model in which agents indulge in YS or TF trading with some probability. It gives a power-law tail with a larger exponent $\nu \simeq 3.7$.

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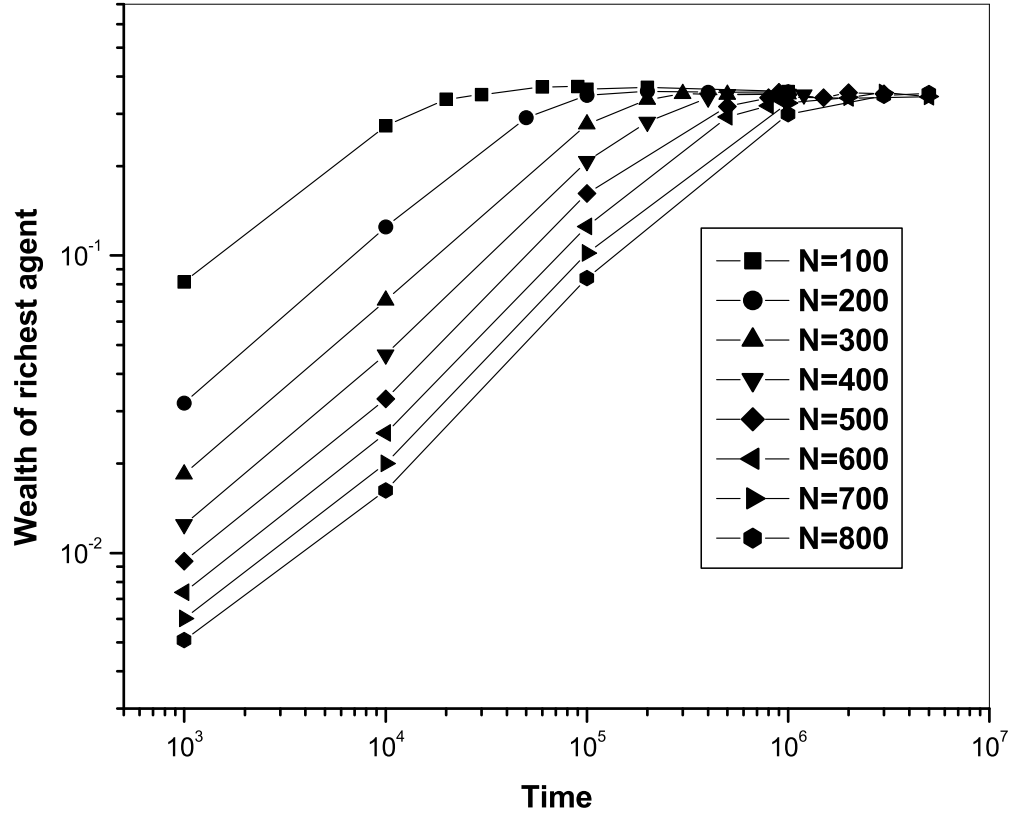[14] M. Rodríguez-Achach and R. Huerta-Quitanilla, Physica A **361** (2006) 309.

Figure 1: Wealth of the richest agent as function of time step for various value of number of agent for model A . We average over 10^3 initial conditions.
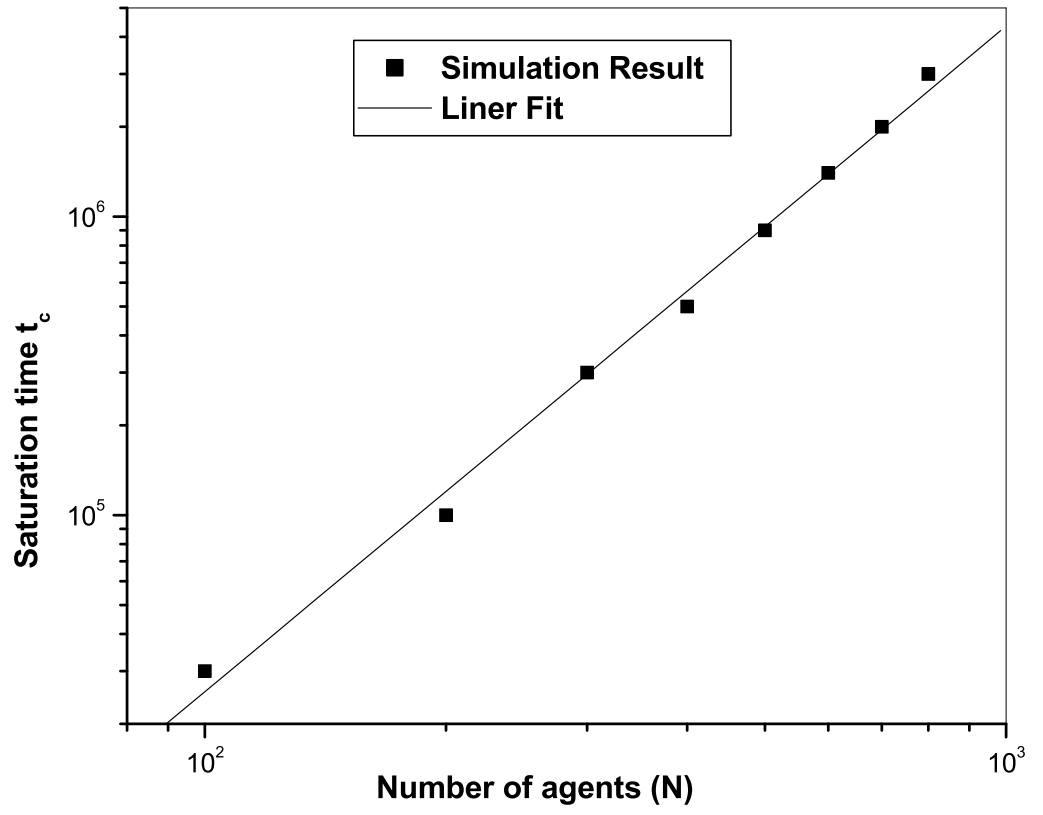
Figure 2: For model A, we plot saturation time t_c as function of number of agent N on logarithmic scale. We average over 10^3 initial conditions.
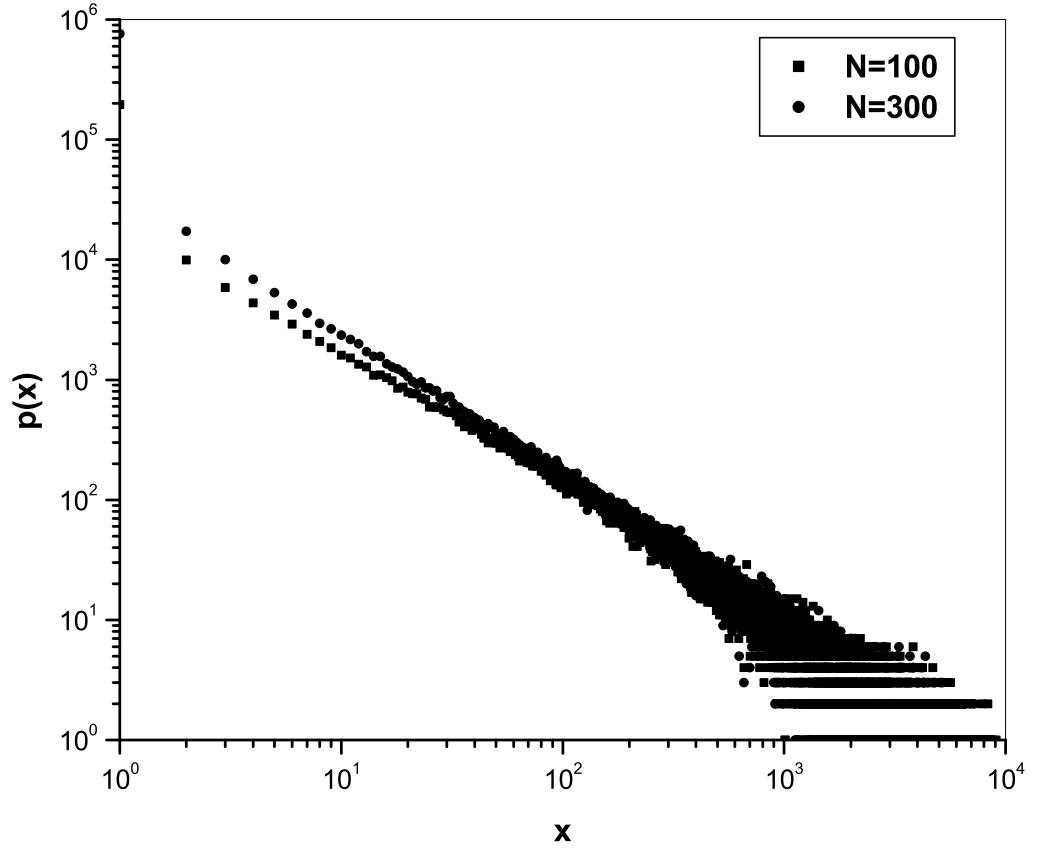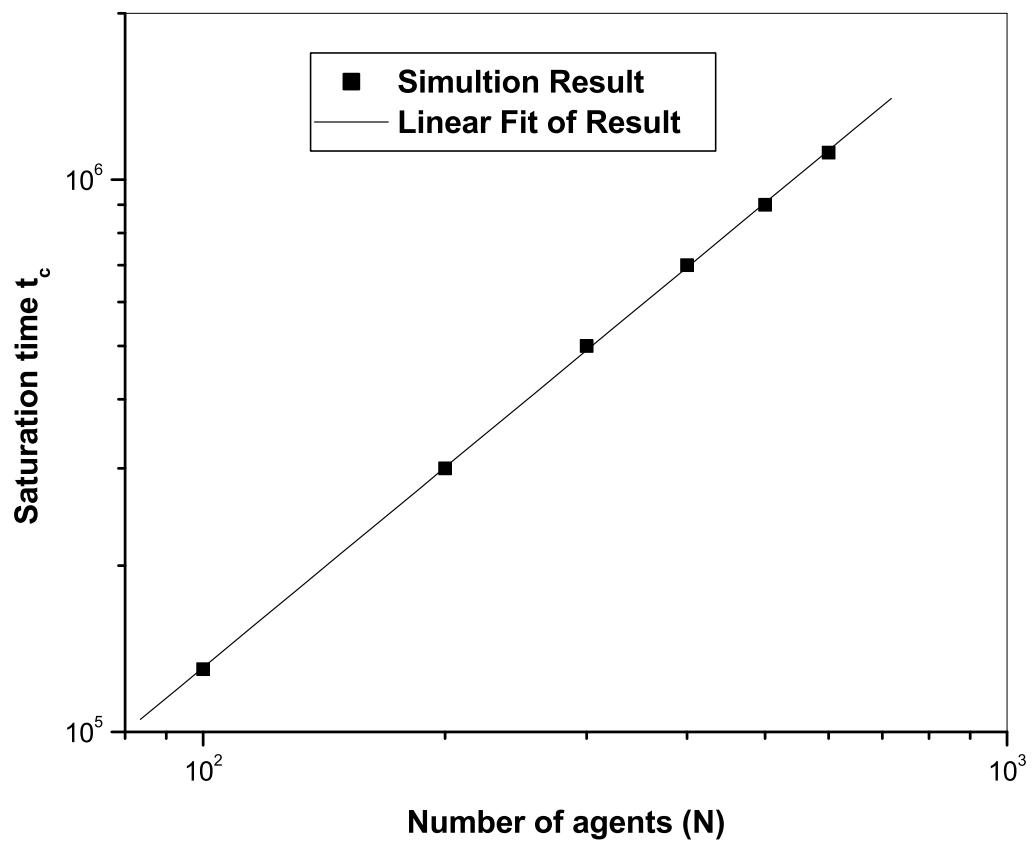
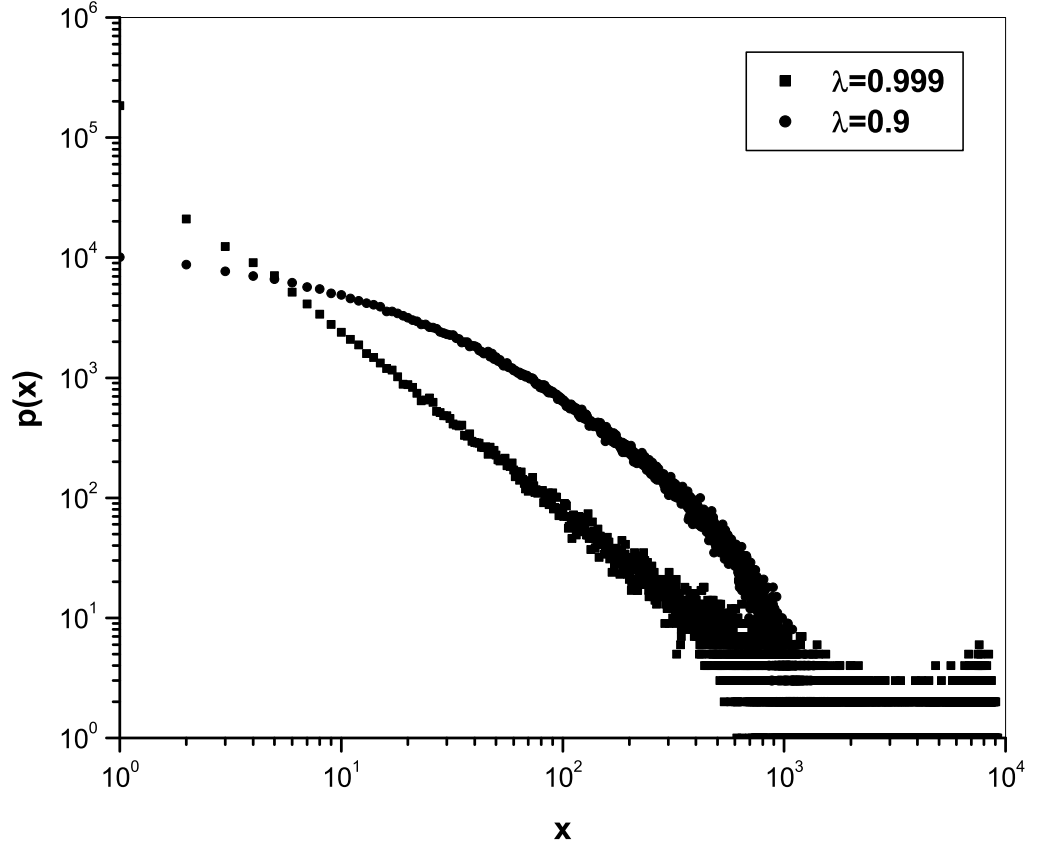Figure 3: Asymptotic wealth distribution for model A. We get a power law with exponent $\nu \simeq 1.1$. Simulations are carried out for $N = 100, 300$. We wait for 10^6 transients and average over 3×10^3 initial conditions[19].
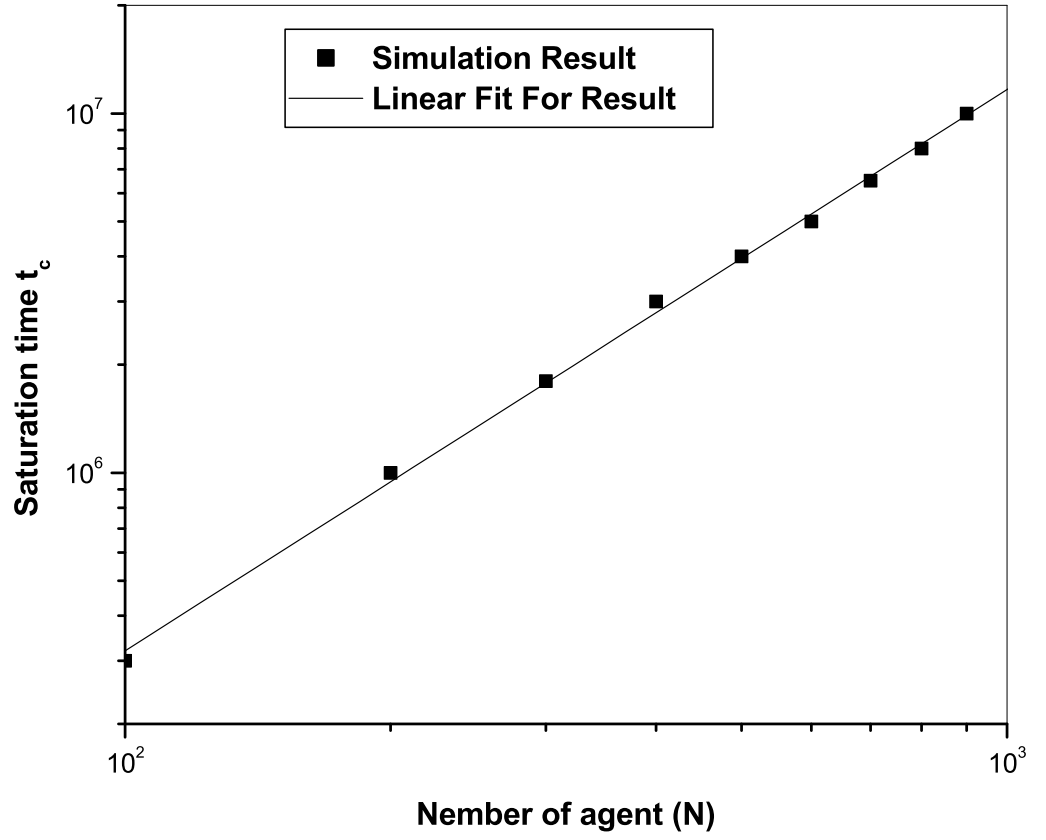
Figure 4: For model B with homogeneous agents, we plot saturation time t_c as a function of number N of agents on logarithmic scale. We average over 10^3 initial conditions.

Figure 5: Asymptotic wealth distribution for different values of λ for model B with homogeneous agents. For $\lambda = 0.999$ we get a power-law tail with exponent $\nu \simeq 1.5$. Simulations are carried out for $N = 100$. We wait for 10^7 transients and average over 3×10^3 initial conditions.

Figure 6: For model B with inhomogeneous agent, we plot saturation time t_c as a function of number of agent N on logarithmic scale. We average over 10^3 initial conditions.
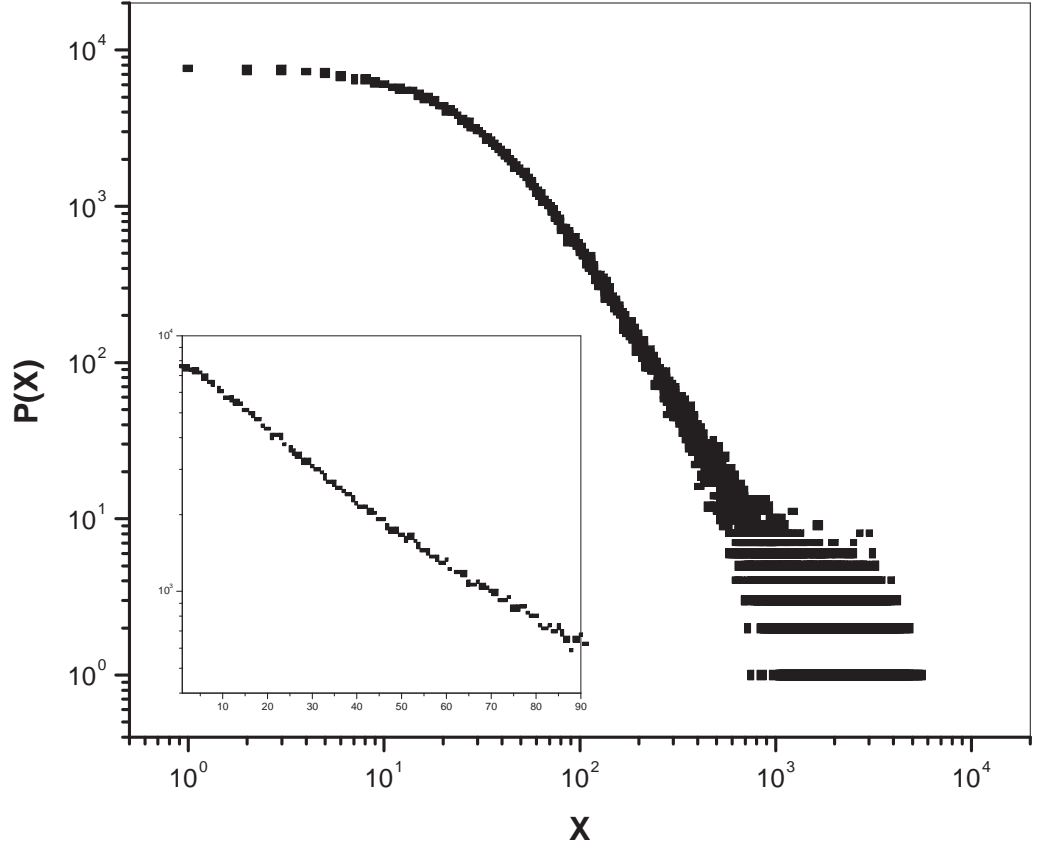
Figure 7: Asymptotic wealth distribution for model B with inhomogeneous agent. we get power law tail with exponent $\nu \simeq 2.0$ at high wealth. Inset: at low wealth, we found an exponential wealth distribution. Simulations are carried out for $N = 100$. We wait for 10^7 transients and average over 3×10^3 initial conditions.
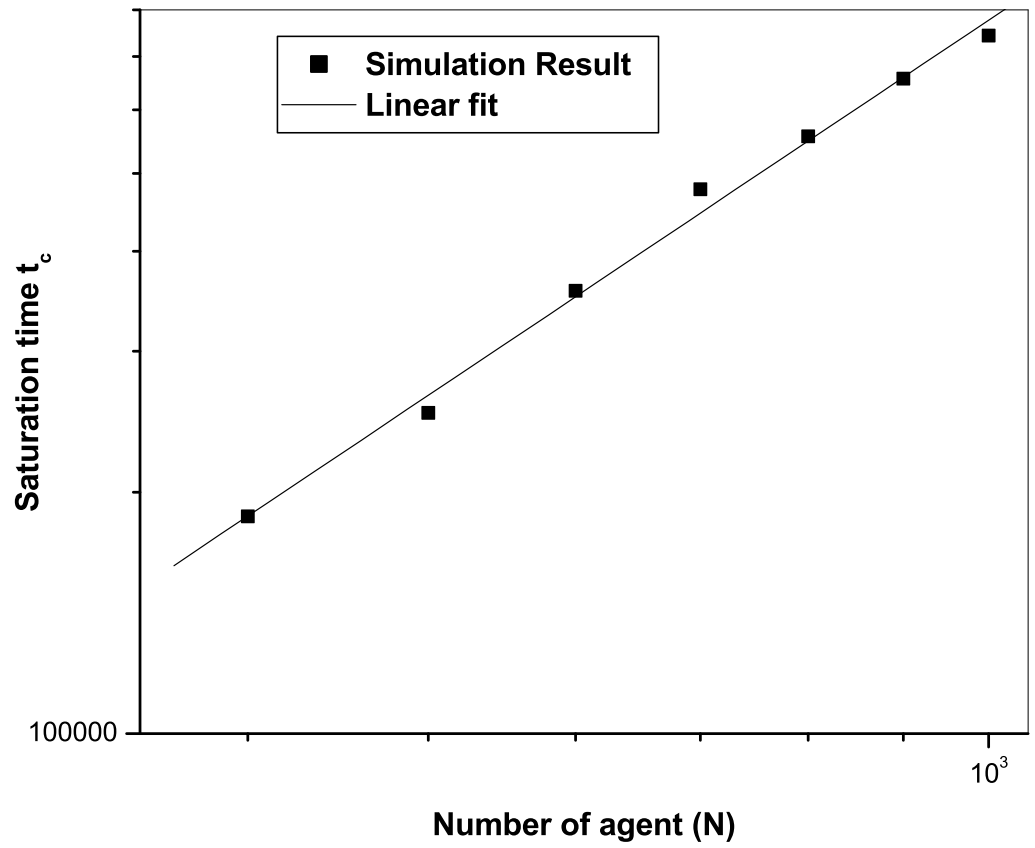
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Figure 8: For model C , we plot saturation time t_c as a function of number of agent N on logarithmic scale. We average over 10^3 initial conditions.
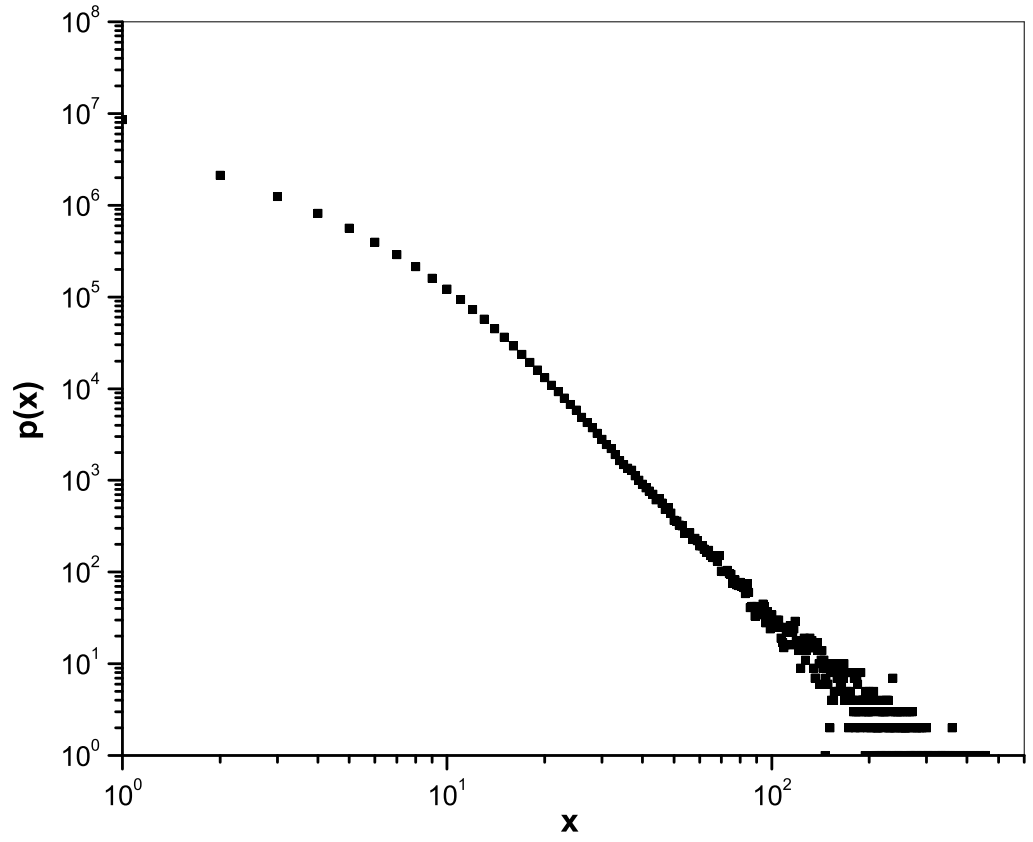
Figure 9: Asymptotic wealth distribution for model C. We get a power law with exponent $\nu \simeq 3.7$. Simulations are carried out for $N = 5000$. We wait for 10^7 transients and average over 3×10^3 initial conditions.

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